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(September 12, 2001)

A new kind of static, spherically symmetric solution to Einstein's equations is described. The solution is characterized by an interior de Sitter region of gravitational vacuum condensate with $p_V = -\rho_V$ and an exterior Schwarzschild geometry of arbitrary total mass M. These are separated by a shell with a small but finite proper thickness ℓ of ultracold matter with the extreme relativistic equation of state $p = \rho$, replacing both the Schwarzschild and de Sitter classical horizons. The new solution has no singularities, no event horizons, and a globally defined timelike Killing field. Its entropy is maximized under small fluctuations and is given by the standard hydrodynamic entropy of the thin shell, which is of order $k_B \ell M c/\hbar$, instead of the Bekenstein-Hawking entropy formula, $S_{BH} = 4\pi k_B GM^2/\hbar c$. Hence unlike black holes, the new solution is thermodynamically stable and has no information paradox. The formation of such a cold ($\sim 1\mu K$) gravitational condensate stellar remnant very likely would require a violent collapse process with an explosive output of energy.

PACS numbers: 04.40.-b, 04.60.-m, 04.70.-s, 73.43.Nq LA-UR-01-5067

Introduction. The vacuum Einstein eqs. of classical general relativity possess a well-known solution for an isolated mass M, with the static, spherically symmetric line element,

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{h(r)} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \,, \qquad (1)$$

where the functions $f(r)$ and $h(r)$ are given by

$$f(r) = h(r) = 1 - \frac{2GM}{r} \,, \qquad (2)$$

in units where $c = 1$. The dynamical singularity of this Schwarzschild metric at $r = 0$ with its infinite tidal forces clearly signals a breakdown of the vacuum Einstein eqs. The kinematical singularity at the Schwarzschild radius or event horizon $R_S = 2GM$ is of a different sort, corresponding to an infinite blue shift of the frequency of an infalling light wave with respect to its frequency far from the black hole. Since the curvature tensor is finite at $r = R_S$, the singularity of the metric (1)-(2) there can be removed by a suitable change of coordinates. Yet whether or not a true event horizon where light itself becomes trapped can be realized in a physical collapse process is open to question.

This question becomes more acute in quantum theory. For when $\hbar \neq 0$, a photon of finite asymptotic frequency ω (even if arbitrarily small) acquires a local energy $E = \hbar\omega f^{-\frac{1}{2}}$, which diverges at $r = R_S$. Since the effective coupling in gravity is $(G/\hbar)^{\frac{1}{2}} E$, the kinematical singularity at R_S is responsible for strong gravitational interactions between elementary quanta as their energy approaches the Planck energy $M_{Pl} = (\hbar/G)^{\frac{1}{2}}$, by which point it is no longer clear that quantum effects on the classical geometry can be safely neglected.

Further, when a massless field such as that of the photon is quantized in the fixed Schwarzschild background, one finds that the black hole radiates these quanta with a thermal spectrum at the asymptotic Hawking temperature $T_H = \hbar/8\pi k_B GM$ [1]. The inverse dependence of T_H on M implies that a black hole in thermal equilibrium with its own Hawking radiation has negative specific heat and is therefore unstable to thermodynamic fluctuations [2]. Energy conservation plus a thermal radiation spectrum imply that the black hole has an enormous entropy, $S_{BH} \simeq 10^{77} k_B (M/M_\odot)^2$ [3], far in excess of a typical stellar progenitor. The associated information paradox has been conjectured to require an alteration in the principles of relativity, or quantum mechanics, or both.

In light of the fundamental challenges quantum black holes pose to current theory, it is reasonable to examine alternatives to the classical view of the event horizon as a harmless kinematical singularity, when the quantum nature of both matter and gravitation is taken into account. Earlier investigations which include the backreaction of the Hawking radiation on the metric suggest that the global geometry may be changed significantly from the classical form (2) [4,5]. In these models the entropy arises entirely from the radiation fluid, and in fact, $S = 4\,\frac{\kappa+1}{7\kappa+1} S_{BH}$, for a fluid with $p = \kappa\rho$, becoming equal to the Bekenstein-Hawking entropy for $\kappa = 1$. However, such self-screened solutions have a negative mass singularity at $r = 0$ and Planckian density radiation at R_S, so that the Einstein eqs. are not reliable in either region.

An interesting proposal for incorporating quantum effects has been made recently in ref. [6], whose authors suggest that the horizon should be replaced by a critical surface of a quantum phase transition, with the interior of the hole replaced by a region with eq. of state, $p = -\rho < 0$. Such an eq. of state is equivalent to a positive cosmological constant term in Einstein's eqs., which does not satisfy the energy condition $\rho+3p \geq 0$, required to prove the classical singularity theorems.

It is our purpose in this Letter to show that an explicit static solution of Einstein's eqs. along these lines can be constructed, with the critical surface of ref. [6] replaced by a thin shell of ultra-relativistic $p = \rho$ matter of the kind considered in ref. [5]. The solution so obtained is free of all singularities and is thermodynamically stable. In fact, the entropy is a local maximum of the hydrodynamic entropy, whose modest value given by (15)-(16) below is easily attainable in a physical collapse process from a progenitor with a comparable mass.

The novel assumption required for this solution to exist is that low energy gravity can undergo a vacuum rearrangement phase transition in the vicinity of $r = R_S$, in which the energy density and eq. of state change. Specifically, since a spatially homogeneous Bose-Einstein condensate (BEC) Ψ couples to Einstein's eqs. in exactly the same way as an effective cosmological term, with eq. of state $\rho_V = -p_V = V(|\Psi|^2)$, the existence of the interior region requires that general considerations of low temperature quantum BEC phase transitions can be extended to gravitation. The effective theory incorporating the low energy effects of quantum anomalies that could give rise to both this interior BEC phase and the $p = \rho$ shell has been presented elsewhere [7]. Here we forego any discussion of the details of the quantum phase transition and present only the solution of Einstein's eqs. with the specified phenomenological eqs. of state.

Solution of Eqs. The eqs. to be solved are the Einstein eqs. for a perfect fluid at rest in the coordinates (1), *viz.*

$$-G^t{}_t = \frac{1}{r^2}\frac{d}{dr}\left[r\,(1-h)\right] = -8\pi G\, T^t{}_t = 8\pi G\rho\,, \qquad \text{(3a)}$$

$$G^r{}_r = \frac{h}{rf}\frac{df}{dr} + \frac{1}{r^2}\,(h-1) = 8\pi G\, T^r{}_r = 8\pi G p\,, \qquad \text{(3b)}$$

together with the conservation eq.,

$$\nabla_a T^a{}_r = \frac{dp}{dr} + \frac{\rho+p}{2f}\,\frac{df}{dr} = 0\,, \qquad (4)$$

which ensures that the other components of the Einstein eqs. are satisfied. These three first order eqs. for the four unknown functions, f, h, ρ, and p become closed when an eq. of state for the fluid, relating p and ρ is specified. Because of the considerations above we allow for three different regions with the three different eqs. of state,

$$\begin{array}{lllll} \text{I.} & \text{Interior}: & 0 \leq r < r_1\,, & \rho = -p\,, & \\ \text{II.} & \text{Shell}: & r_1 < r < r_2\,, & \rho = +p\,, & \qquad (5)\\ \text{III.} & \text{Exterior}: & r_2 < r\,, & \rho = p = 0\,. & \end{array}$$

At the interfaces $r = r_1$ and $r = r_2$, we require the metric coefficients r, f and h to be continuous, although the first derivatives of f and h must be discontinuous from the first order eqs. (3) and (4).

In the interior region $\rho = -p$ is a constant from (4). We call this constant $\rho_V = 3H_0^2/8\pi G$. If we require that $f(0) = h(0) = 1$ so that the origin is locally flat and free of any mass singularity, then the interior region is uniquely determined to be a region of de Sitter spacetime in static coordinates, *i.e.*

$$\text{I.} \qquad f(r) = h(r) = 1 - H_0^2\, r^2\,, \qquad 0 \leq r \leq r_1\,. \qquad (6)$$

Likewise the unique solution in the exterior vacuum region which approaches flat spacetime as $r \to \infty$ and is free of any cosmological singularity is a region of Schwarzschild spacetime (2), *viz.*

$$\text{III.} \qquad f(r) = h(r) = 1 - \frac{2GM}{r}\,, \qquad r_2 \leq r\,. \qquad (7)$$

The two integration constants H_0 and M are arbitrary.

The only non-vacuum region is region II. Let us define the dimensionless variable w by $w \equiv 8\pi G r^2 p$, so that eqs. (3)-(4) with $p = \rho$ may be put into the form,

$$\frac{dr}{r} = \frac{dh}{1-w-h}\,, \qquad \text{(8a)}$$

$$\frac{dh}{h} = -\frac{1-w-h}{1+w-3h}\,\frac{dw}{w}\,. \qquad \text{(8b)}$$

The first of these eqs. is equivalent to the definition of the (rescaled) Tolman mass function μ by $h = 1 - \mu/r$ and $d\mu(r) \equiv 2\,G\,dm(r) = 8\pi G\,\rho r^2\,dr = w\,dr$ within the shell. The second eq. can be solved only numerically in general. However, it is possible to obtain an analytic solution in the thin shell limit, $0 < h \ll 1$, for in this limit we can set h to zero on the right side of (8b) to leading order, and integrate it immediately to obtain

$$h \equiv 1 - \frac{\mu}{r} \simeq \epsilon\, \frac{(1+w)^2}{w} \ll 1\,, \qquad (9)$$

in region II, where ϵ is an integration constant. Because of the condition $h \ll 1$ we require $\epsilon \ll 1$. Making use of eqs. (8) and (9) we have

$$\frac{dr}{r} \simeq -\epsilon\, dw\, \frac{(1+w)}{w^2}\,, \qquad (10)$$

which gives to first order in ϵ,

$$r \simeq r_1 \left[1 - \epsilon \ln\left(\frac{w}{w_1}\right) + \epsilon\left(\frac{1}{w} - \frac{1}{w_1}\right)\right]\,. \qquad (11)$$

Because of the approximation $\epsilon \ll 1$, the radius r hardly changes within region II, and dr is of order $\epsilon\, dw$. Since $p = \rho$, eq. (4) requires $pf \propto wf/r^2$ to be a constant, so that the final unknown function f is given by $f = (r/r_1)^2(w_1/w)f(r_1) \simeq (w_1/w)f(r_1)$ for small ϵ.

Now requiring continuity of the metric coefficients f and h at both r_1 and r_2 gives the conditions,

$$f(r_1) = h(r_1) = 1 - H_0^2 r_1^2 \simeq \epsilon\, \frac{(1+w_1)^2}{w_1}\,, \qquad \text{(12a)}$$

$$f(r_2) = 1 - \frac{2GM}{r_2} = \left(\frac{r_2}{r_1}\right)^2 \frac{w_1}{w_2} f(r_1) \simeq \frac{w_1}{w_2} f(r_1)\,, \quad \text{(12b)}$$

$$h(r_2) = 1 - \frac{2GM}{r_2} \simeq \epsilon\, \frac{(1+w_2)^2}{w_2}\,. \qquad \text{(12c)}$$

Together with r_2/r_1 from (11) this gives four relations between the five integration constants $(r_1, r_2, w_1, w_2, \epsilon)$. Hence the first four can be eliminated in favor of H_0, M and $\epsilon \ll 1$, and we have a three parameter family of static solutions. Assuming that (r_1, r_2, w_1, w_2) remain finite as $\epsilon \to 0$, *i.e.* are of order ϵ^0, we obtain from (11) and (12) that $r_1 \simeq H_0^{-1} \simeq r_2 \simeq 2GM$, but

$$1 - H_0^2 r_1^2 \sim 1 - \frac{2GM}{r_2} \sim \frac{w_1}{w_2} - 1 \sim \sqrt{\frac{r_2}{r_1} - 1} \sim \epsilon\,. \quad (13)$$

If $\epsilon > 0$ then both f and h are of order ϵ and approximately constant in region II, but are nowhere vanishing. Hence there is no event horizon and ∂_t is a globally defined timelike Killing vector of the static solution.

Principal Features. The physical meaning of $\epsilon \ll 1$ is that $\epsilon^{-\frac{1}{2}}$ is the order of the very large but finite blue shift a photon experiences in falling into the shell from infinity. The proper thickness of the shell,

$$\ell = \int_{r_1}^{r_2} dr\, h^{-\frac{1}{2}} \simeq R_S \epsilon^{\frac{1}{2}} \int_{w_2}^{w_1} dw\, w^{-\frac{3}{2}} \sim \epsilon^{\frac{3}{2}} R_S \qquad (14)$$

is small compared to R_S, since $dw \sim w_1 - w_2 \sim \epsilon$ and hence $dr \sim r_2 - r_1 \sim \epsilon^2$ from (10) or (11). Although the length scale ℓ is arbitrary here, presumably it is fixed by microscopic physics and not greater than the Planck length by more than a few orders of magnitude. If we assume that ℓ is a fixed multiple of the Planck length and independent of M, then $\epsilon \sim (\ell/GM)^{\frac{2}{3}} \sim (M_{Pl}/M)^{\frac{2}{3}} \simeq 10^{-25\pm 1}$ for a solar mass object, which certainly justifies the small ϵ approximation in this case.

The entropy of the shell is obtained from the eq. of state, $p = \rho = (a^2/8\pi G)(k_B T/\hbar)^2$, where we have introduced G for dimensional reasons so that the constant a^2 is dimensionless. By the standard thermodynamic relation, $Ts = p + \rho$ for a relativistic fluid with zero chemical potential, so the local specific entropy density $s(r) = a^2 k_B^2 T(r)/4\pi\hbar^2 G = a(k_B/\hbar)(p/2\pi G)^{\frac{1}{2}}$ for local temperature $T(r)$. Thus $s = (a k_B/4\pi\hbar G r) w^{\frac{1}{2}}$ and the entropy of the fluid within the shell is

$$S = 4\pi \int_{r_1}^{r_2} s\, r^2\, dr\, h^{-\frac{1}{2}} \simeq \frac{a k_B R_S^2}{\hbar G}\, \epsilon^{\frac{1}{2}} \;\ln\left(\frac{w_1}{w_2}\right)\,. \quad (15)$$

From (13) and (14) this is of order

$$S \sim a\, k_B\, \frac{M}{\hbar}\, R_S \epsilon^{\frac{3}{2}} \sim a\, k_B \frac{M\ell}{\hbar} \ll S_{BH}\,. \qquad (16)$$

Since the entropy of the interior vacuum condensate has $p_V = -\rho_V$, Ts vanishes in the interior region I, as well as the exterior vacuum region III. This is as expected for a BEC described by a macroscopic single quantum state. The entropy of the entire quasi-black hole (QBH) is given then by (15) or (16), which is of order $10^{38} k_B (\ell/L_{Pl})$ for a solar mass object. This is some 38 orders of magnitude lower than the Bekenstein-Hawking entropy for the same mass M, and 20 ± 1 orders of magnitude lower than a typical stellar progenitor which have entropies in the range of $10^{57} k_B$ to $10^{59} k_B$ for $M/m_N \sim 10^{57}$ nucleons. Since w is of order unity in the shell while $r \simeq R_S$, the *local* temperature of the fluid within the shell is of order $T_H \sim 1\mu K$. However because of the global timelike Killing field and absence of event horizon, the shell does not emit Hawking radiation. The gravitational condensate remnant is both ultracold and completely dark.

The extremely cold radiation fluid in the shell is confined to region II by the surface tensions at the timelike interfaces r_1 and r_2. These arise from the pressure discontinuities, $\Delta p_1 = H_0^2\,(3 + w_1)/8\pi G$ and $\Delta p_2 = -w_2/32\pi G^3 M^2$, and are calculable by the Israel junction conditions [8]. We find that the non-zero angular components of the surface tension are

$$\sigma^\theta{}_\theta = \sigma^\phi{}_\phi = \frac{H_0}{8\pi G}\left(\frac{w_1}{\epsilon}\right)^{\frac{1}{2}}, \qquad \text{(17a)}$$

$$\sigma^\theta{}_\theta = \sigma^\phi{}_\phi = -\frac{1}{32\pi G^2 M^2}\frac{w_2}{(1+w_2)}\left(\frac{w_2}{\epsilon}\right)^{\frac{1}{2}}. \qquad \text{(17b)}$$

at r_1 and r_2 respectively. The signs correspond to the inner surface at r_1 exerting an outward force and the outer surface at r_2 exerting an inward force, *i.e.* both surface tensions exert a confining pressure on the shell region II. The time component $\sigma^t{}_t = 0$, corresponding to vanishing contribution to the Tolman mass function $m(r)$ at the two interfaces. Since $\epsilon^{-\frac{1}{2}} \sim (M/M_{pl})^{\frac{1}{3}}$, the surface tensions (17) are of order $M^{-\frac{2}{3}}$ and far from Planckian. Hence the matching of the metric at the phase interfaces r_1 and r_2, analogous to that across stationary shocks in hydrodynamics, should be reliable. Resolving the interfaces will require going beyond Einstein's eqs. to a more microscopic description of the quantum phase transition.

The energy within the shell (as measured at infinity),

$$E_{\rm II} = 4\pi \int_{r_1}^{r_2} \rho\, r^2 dr \simeq \epsilon M \int_{w_2}^{w_1} dw \sim \epsilon^2 M \propto M^{-\frac{1}{3}} \quad (18)$$

is also very small. Hence essentially all the mass of the object comes from the energy density of the vacuum condensate in the interior, even though the shell is responsible for all of its entropy.

Stability. In order to be a physically realizable endpoint of gravitational collapse, any quasi-black hole candidate must be stable [9]. Since only the region II with a 'normal' fluid component is subject to fluctuations and its heat capacity is positive, it is clear on physical grounds that the solution is thermodynamically stable. The analysis of stability within the canonical ensemble is complicated by the existence of the three parameter family (H_0, ℓ, M), and the ability of the two interfaces to move freely, transferring heat to/from region II by $p\,dV$ work. It is therefore simplest to work in the microcanonical ensemble where all three parameters and the positions of the interfaces are fixed. Then stability may be demonstrated by showing that the entropy functional,

$$S = \frac{ak_{_B}}{\hbar G}\int_{r_1}^{r_2} r\,dr\,\left(\frac{d\mu}{dr}\right)^{\frac{1}{2}}\left(1-\frac{\mu(r)}{r}\right)^{-\frac{1}{2}}\,, \tag{19}$$

is maximized under all variations of $\mu(r)$ in region II with the endpoints (r_1, r_2), or equivalently (w_1, w_2) fixed.

The first variation of this functional with the endpoints r_1 and r_2 fixed vanishes, *i.e.* $\delta S = 0$ by the Einstein eqs. (3) for a static, spherically symmetric star. Thus any solution of eqs. (3)-(4) is guaranteed to be an extremum of S [10]. This is also consistent with regarding Einstein's eqs. as a form of hydrodynamics, strictly valid only for the long wavelength, gapless excitations in gravity.

The second variation of (19) is

$$\delta^2 S = \frac{ak_{_B}}{4\hbar G}\int_{r_1}^{r_2} r\,dr\,\left(\frac{d\mu}{dr}\right)^{-\frac{3}{2}} h^{-\frac{1}{2}}\,\times \left\{-\left[\frac{d(\delta\mu)}{dr}\right]^2 + \frac{(\delta\mu)^2}{r^2h^2}\frac{d\mu}{dr}\left(1+\frac{d\mu}{dr}\right)\right\}\,, \tag{20}$$

when evaluated on the solution. Associated with this quadratic form in $\delta\mu$ is a second order linear differential operator of the Sturm-Liouville type, *viz.*

$$\frac{d}{dr}\left\{r\left(\frac{d\mu}{dr}\right)^{-\frac{3}{2}} h^{-\frac{1}{2}}\frac{d\chi}{dr}\right\} + \frac{h^{-\frac{5}{2}}}{r}\left(\frac{d\mu}{dr}\right)^{\frac{1}{2}}\left(1+\frac{d\mu}{dr}\right)\chi \equiv \mathcal{L}\chi\,. \tag{21}$$

This operator possesses two solutions satisfying $\mathcal{L}\chi_{_0} = 0$, obtained by variation of the classical solution, $\mu(r; r_1, r_2)$ with respect to the parameters (r_1, r_2). Since these correspond to varying the positions of the interfaces, $\chi_{_0}$ does not vanish at (r_1, r_2) and neither function is a true zero mode. However, we may set $\delta\mu \equiv \chi_{_0}\,\psi$, where ψ does vanish at the endpoints and insert this into the second variation (20). Integrating by parts, using the vanishing of $\delta\mu$ at the endpoints and $\mathcal{L}\chi_{_0} = 0$ gives

$$\delta^2 S = -\frac{ak_{_B}}{4\hbar G}\int_{r_1}^{r_2} r\,dr\,\left(\frac{d\mu}{dr}\right)^{-\frac{3}{2}} h^{-\frac{1}{2}}\chi_{_0}^2\left(\frac{d\psi}{dr}\right)^2 < 0\,. \tag{22}$$

Thus the entropy of the solution is a maximum with respect to all radial variations that vanish at the endpoints, *i.e.* those with fixed total energy.

By changing variables from r to w and using the explicit solution (9)-(10) it is readily verified that one solution to $\mathcal{L}\chi_{_0} = 0$ is $\chi_{_0} = 1 - w$, from which the second linearly independent solution $(1-w)\ln w + 4$ may be obtained. Substituting then *e.g.* $\delta\mu = (1-w)\psi(w)$ in (20) brings the second variation of the entropy into the form (22) which is explicitly negative definite. Since deformations with non-zero angular dependence decrease the entropy even further, stability under radial variations is sufficient to demonstrate that the solution is completely stable to small deformations. Hence it is a viable alternative for the stable endpoint of gravitational collapse.

Observational Consequences. Although we have presented only a static spherically symmetric solution, it is clear on physical grounds that axisymmetric rotating solutions should exist as well. These are presumably characterized by a finite density of vortices of normal phase penetrating the condensate core. Such rotating gravitational vacuum condensate stars, dark 'grava(c)stars' are candidates for the stable remnants of stellar evolution for stars exceeding the Chandrasekhar limit. Since the entropy of these objects is some 20 orders of magnitude smaller than that of a typical stellar progenitor, a violent process of entropy shedding, as in a supernova, is needed to produce a cold gravastar remnant.

Explosive bursts in which a finite fraction of atoms are ejected have been observed in attractive BEC's in the laboratory [11]. The remnants are left in an excited oscillatory state afterwards. In the present case the shell with the maximally stiff eq. of state $p = \rho$ would be expected to produce an outgoing shock front which would accelerate everything in its vicinity to ultra-relativistic energies, producing ultrahigh energy particles, gamma rays and gravitational radiation. Such stellar hypernovae may be the engines of gamma ray bursters and active galactic nuclei, as well as efficient cosmic ray accelerators. The spectra of gravitational radiation should bear the imprint of the fundamental frequencies of vibration of the gravastar. Finally, we note that the interior de Sitter region with $p = -\rho$ may be interpreted also as a cosmological spacetime, with the horizon of the expanding universe replaced by a quantum phase interface.

Research of P. O. M. supported in part by an NSF grant.

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